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Bnach Contraction Method for Solving Telegraph Equation

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Abstract
In this research paper, we employ the semi-analytical method known as the Banach contraction method (BCM) to obtain an approximate solution for the telegraph equation. This method yields reliable results in the form of analytical approximations, making it a highly dependable alternative for finding analytical solutions to the telegraph equation. The behavior of the equation is evaluated through computer simulations using MATLAB code.
Keywords: Telegraph equation, Integral equation, Banach contraction method.

طريقة الانكماش لبناخ لحل معادلة التلغراف

أميرة شعيب، سهيلة ابريش

قسم الرياضيات - كلية التربية الزاوية، الزاوية -ليبيا

الملخص
في هذه الورقة البحثية، نستخدم الطريقة الشبه تحليلية و هي طريقة الانكماش لباناخ للحصول على حل تقريبي لمعادلة التلغراف، توفر هذه الطريقة نتائج موثوقة في شكل تقريبات تحليلية، مما يجعلها بديلا موثوقا لإيجاد حلول تحليلية لمعادلة التلغراف ، يتم تقييم سلوك المعادلة من خلال المحاكاة الحاسوبية باستخدام **MATLAB.**
الكلمات المفتاحية: معادلة التلغراف المعادلة التكاملية، طريقة الانكماش لبناخ.

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1. Introduction

The telegraph equation was originaly introduced by Kirchhoff in 1857, but it was first examined by Poincaré in 1893. The telegrapher's equation exhibits characteristics of both wave motion and diffusion. It has numerous important applications in industrial processes, particularly in the field of communication systems. Several methods have been employed to solve the telegraph equation, including the homotopy perturbation method (HPM) [1], the Daftardar-Gejji-Jafaris method (DGJ) [2], the Laplace transform (LT) [3], the q-homotopy analysis transform method (q-HATM) [4], and the reduced differential transform method (RDTM) [5]. In a previous study [6], Sehgal and Bharucha-Reid proposed a novel semi-analytical iterative method known as the Banach Contraction method (BCM). The BCM method, which builds upon the Picard method, has been successfully applied to solve a wide range of differential and integral equations [7,8]. In this investigation, we focus on the telegraph equation and its analysis using the BCM method. We consider the telegraph equation

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + cu. \qquad (1)$$

With the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x). \qquad (2)$$

where $a, b\ and\ c$ denote positive constant, also $f(x)\ and\ g(x)$ are known continues functions.

In this paper, we will implement the BCM to find approximate solutions for telegraph equation.

2. Banach contraction method

Definition 2.1 [9]: Let (X, d) be a metric space, then a mapping $f: X \to X$,

(a) A point $x \in X$is called a fixed point of f if $x = f(x)$.

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(b)f is called contraction if there exists a fixed constant $k < 1$ such that

$$d\big(f(x), f(y)\big) \leq kd(x, y), for\ all\ x, y\ \in X.$$

A contraction mapping is also known as Banach contraction.

Theorem 2.2 (Bnach contraction principle) [10]:

Let (X, d) be a complete metric space and $f: X \to X$ be a contraction mapping. Then f has a unique fixed point x_0 and for each $x \in X$, we have

$$f^n(x) = x_0\ . \quad (3)$$

Moreover, for each $x \in X$, we have

$$d(f^n(x), x_0)\ \leq \frac{k^n}{1-k} d(f(x), x).$$

Theorem 2.3 [10]:

Let $f(x, u)$ be a continuous function on $[a, b] \times [c, d]$ such that f is Lipschitz with respect to u, that is there exists $k > 0$, such that $|f(x, u) - f(x, v)| \leq k\ |u - v|$, for all $u, v\ \in [c, d]$ and for $x \in [a, b]$.

Banach contraction method 2.4:

Let us consider the general functional equation [9],

$$u = N(u) + f \quad (4)$$

Where $N(u)$ is a nonlinear operator and f is a known function. Define successive approximations as

$$u_0 = f,$$
$$u_1 = u_0 + N(u_0),$$
$$u_2 = u_0 + N(u_1),$$
$$u_3 = u_0 + N(u_2),$$
$$\vdots$$
$$u_n = u_0 + N(u_{n-1}), \qquad n = 1,2,3, \dots . \quad (5)$$

If N^k is contraction for operator some positive integer k, then $N(u)$ has a unique fixed point and hence the sequence defined by (5) is convergent in view of theorem (2.2) and the solution of eq. (3) is given by $u = u_n$.

3. Application of the method to the telegraph equation

To solve the telegraph equation using the BCM method we first convert the equation into an integral equation.
Initially we specify the conditions of the telegraph equation which're the given values of the variables at the starting point. These initial conditions are then utilized to transform the equation into an integral equation.
A Volterra equation is an equation that establishes a relationship, between the desired function and the integral of the equation. By applying integration principles to the equation, we can convert it into an equation. This particular equation involves a function that needs to be determined.
Subsequently we employ the BCM method to solve this resulting equation. The BCM method relies on iterations to estimate a value for our unknown function. We continue this iteration process until we obtain an approximation, for our desired solution. according to the initial condition we can converted to the Volterra integral equation

$$u(x,t) = f(x,t) + N(u) \tag{6}$$

We start with the telegraph equation (1)

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + cu. \tag{7}$$

We obtain

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - cu. \tag{8}$$

By integrating both sides from 0 to t and substitute for the initial conditions of the equation, we have

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$$a\frac{\partial u}{\partial t} = ag(x) + bf(x) - bu + \int_0^t \left(\frac{\partial^2 u}{\partial x^2} - cu\right)dt. \quad (9)$$

Again, by integrating both sides from 0 to t, substitute for the initial conditions of the equation, reducing multiple integrals to a single integral and division by a , it leads the following integral equation

$$u(x,t) = f(x) + \big(g(x) + \alpha f(x)\big)t + \int_0^t \frac{1}{a}(t-s)\left(\frac{\partial^2 u}{\partial x^2} - cu\right) - \alpha u)dt. \quad (10)$$

Where $\alpha = \frac{b}{a}$,

Let $$F(x,t) = f(x) + \big(g(x) + \alpha f(x)\big)t\,. \quad (11)$$

And $$N(u) = \int_0^t \left(\frac{1}{a}(t-s)\left(\frac{\partial^2 u}{\partial x^2} - cu\right) - \alpha u\right)dt. \quad (12)$$

Implementation of the BCM algorithm gives

$$\begin{aligned} u_0(x,t) &= F(x,t) \\ u_1(x,t) &= u_0(x,t) + N(u_0) \\ u_2(x,t) &= u_0(x,t) + N(u_1) \\ &:: \\ u_n(x,t) &= u_0(x,t) + N(u_{n-1}) \end{aligned} \quad (13)$$

The BCM admits the use of

$$u(x,t) = u_n(x,t)$$

In other words, the functional (13) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

4. Numerical Examples

To verify and validate the efficiency and reliability of the BCM method, we give several examples. These examples are chosen from [5,7] also graphs of comparison between exact and approximate solution. Now we began with the following examples.

Example 4.1

Consider the linear telegraph equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u \, . \tag{14}$$

Subject to initial conditions

$$u(x,0) = e^x, u_t(x,0) = -2e^x. \tag{15}$$

Appling the BCM method to (14),we obtain the following

$$u(x,t) = e^x + \int_0^t \quad ((t-s)(u_{xx} - u) - 2u)\, ds, \tag{16}$$

Let $F(x,t) = e^x$ and $N(u) = \int_0^t \quad \big((t-s)(u_{xx} - u) - 2u\big)ds.$

$$u_0 = e^x \, ,$$

$$u_1 = u_0 + N(u_0),$$

$$u_1 = e^x + \int_0^t \quad \big((t-s)(u_{0xx} - u_0) - 2u_0\big)ds.$$

$$u_1 = e^x - 2\, te^x.$$

$$u_2 = e^x + \int_0^t \quad \big((t-s)(u_{1xx} - u_1) - 2u_1\big)ds.$$

$$u_2 = e^x - 2\, te^x + \frac{4}{2}t^2 e^x.$$

$$u_3 = e^x + \int_0^t \quad \big((t-s)(u_{2xx} - u_2) - 2u_2\big)ds.$$

$$u_3 = e^x - 2\, te^x + \frac{4}{2}t^2 e^x - \frac{8}{2 \cdot 3}t^3 e^x.$$

$$u_4 = e^x + \int_0^t \quad \big((t-s)(u_{3xx} - u_3) - 2u_3\big)ds.$$

$$u_4 = e^x - 2\, te^x + \frac{4}{2}t^2 e^x - \frac{8}{2 \cdot 3}t^3 e^x + \frac{16}{2 \cdot 3 \cdot 4}t^4 e^x.$$

$$\vdots$$

$$u_n = e^x - 2\, te^x + \frac{4}{2!}t^2 e^x - \frac{8}{3!}t^3 e^x + \cdots + \frac{(-2t)^n}{n!}e^x. \tag{17}$$

The BCM admits the use of

$$u(x,t) = u_n \ ,$$

That gives the exact solution by

$$u(x,t) = e^{x-2t} \tag{18}$$

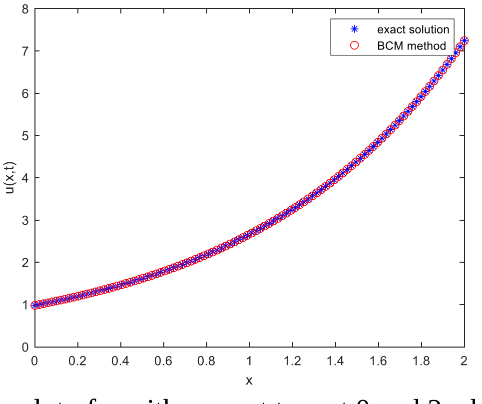

Figure 1: plot of u with respect to x at 0 and 2 where t=1.

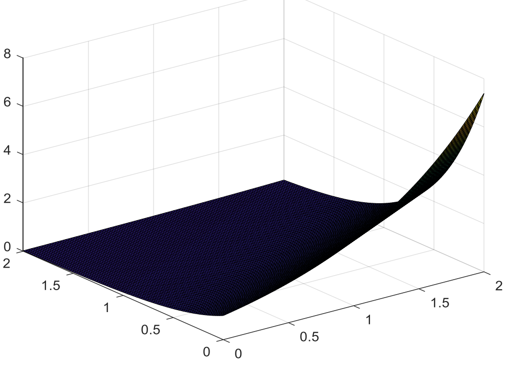

Figure 2: Depiction of solution u in the domain $x \in [0,2]$ and $t \in [0,1]$.

Example 4. 2

Consider the linear telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u \quad (19)$$

With the boundary conditions

$$u(0,t) = 1 + e^{-2t}, u_x(0,t) = 2. \quad (20)$$

Where, the exact solution is

$$u(x,t) = e^{-2t} + e^{2x} \quad (21)$$

Integrate both sides of Eq. (19) twice from 0 to x and using the boundary conditions, we get

$$u(x,t) = 1 + 2x + e^{-2t} + \int_0^x \int_0^x \left(u_{tt}(s,t) + 4u_t(s,t) + 4u(s,t)\right) dsds. \quad (22)$$

By reducing the double integration to single, we have

$$u(x,t) = 1 + 2x + e^{-2t} + \int_0^x (x-s)\left(u_{tt}(s,t) + 4u_t(s,t) + 4u(s,t)\right) ds \quad (23)$$

And

$$N(u) = \int_0^x (x-s)\left(u_{tt}(s,t) + 4u_t(s,t) + 4u(s,t)\right) ds \quad (24)$$

By implementing the BCM, we obtain the following

$$u_0(x,t) = 1 + 2x + e^{-2t}.$$

$$u_1(x,t) = u_0 + N(u_0).$$

$$u_1(x,t) = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{2} + \frac{16x^4}{4} + \frac{32x^5}{4} + e^{-2t}$$

$$u_2(x,t) = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{2} + \frac{16x^4}{4} + \frac{32x^5}{4} + \frac{64x^6}{8} + \frac{128x^7}{8} + e^{-2t}.$$

$$\vdots$$

$$u_n(x,t) = e^{-2t} + 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{2} + \frac{16x^4}{4} + \frac{32x^5}{4} + \frac{64x^6}{8} + \frac{128x^7}{8} + \cdots \quad (25)$$

This close from

$$u(x,t) = e^{-2t} + e^{2x} \tag{26}$$

Which is the exact solution.

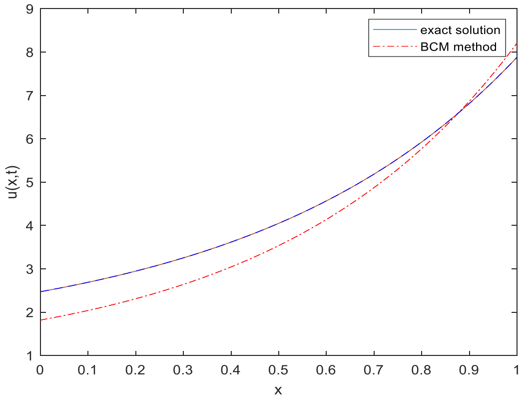

Figure 3: plot of u with respect to x at 0 and 1 where t=1

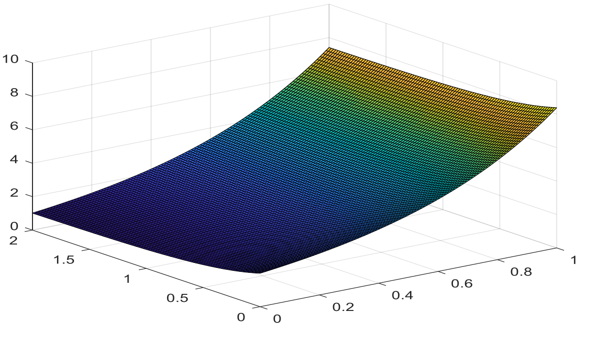

Figure 4: Depiction of solution u in the domain $x \in [0,2]$ and $t \in [0,1]$.

Example 4.3

Consider the nonlinear telegraph equation

$$u_{xx} + x^2 + t - 1 = u_{tt} + u_t + u \tag{27}$$

With the initial condition

$$u(x,0) = x^2, u_t(x,0) = 1. \tag{28}$$

Where, the exact solution is

$$u(x,t) = x^2 + t. \tag{29}$$

Integrate both sides of Eq. (27) twice from 0 to t and using the initial conditions, we get

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!} + \int_0^t \big((t-s)(u_{xx} - u) - u\big)ds \tag{30}$$

Let $F(x,t) = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!}$,

And $N(u) = \int_0^t \big((t-s)(u_{xx} - u) - u\big)ds.$

$$u_0 = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!},$$

$$u_1 = u_0 + N(u_0),$$

$$u_1 = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!} + \int_0^t \big((t-s)(u_{0xx} - u_0) - u_0\big)ds.$$

$$u_1 = \left(1 - t^2 - t^3 - \frac{t^4}{4}\right)x^2 + t - \frac{t^2}{2} + \frac{7t^3}{6} + \frac{7t^4}{12} - \frac{t^5}{12}.$$

$$u_2 = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!} + \int_0^t \big((t-s)(u_{1xx} - u_1) - u_1\big)ds.$$

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$$u_2 = \left(1 - t^4 - 2t^5 - \frac{3t^6}{4}\right)x^2 + t - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{12} - \frac{17t^5}{6} + \frac{37t^6}{8} + \frac{t^7}{24}.$$

$$u_3 = x^2\left(1 + t + \frac{t^2}{2}\right) + t - \frac{t^2}{2} + \frac{t^3}{3!} + \int_0^t \left((t-s)(u_{2xx} - u_2) - u_2\right)ds.$$

u_3 = t - (t^2x^2)/2 + (t^3x^2)/6 + (t^4x^2)/8 + (t^5x^2)/40 + (t^6x^2)/720 + $x^2(t^2/2$ + t + 1) - tx^2- (5t^4)/24 - (3t^5)/40 - t^6/240 + t^7*/5040*.

⋮

$$u_n = x^2\left(1 + t + \frac{t^2}{2}\right) + \left(1 - t^2 - t^3 - \frac{t^4}{4}\right)x^2 + \left(1 - t^4 - 2t^5 - \frac{3t^6}{2} + \frac{t^7}{2} - \frac{t^8}{16}\right)x^2 + 2t - 2\frac{t^2}{2} + 2\frac{t^3}{3!} + \frac{t^4}{12} + \frac{17\,t^5}{6} + \frac{37t^6}{8} + \frac{55t^7}{24} + \frac{5t^8}{16} + \frac{t^9}{48} + \cdots.$$

The BCM admits the use of

$u(x,t) = u_n$,

That gives the exact solution by

$$u(x,t) = x^2 + t.$$

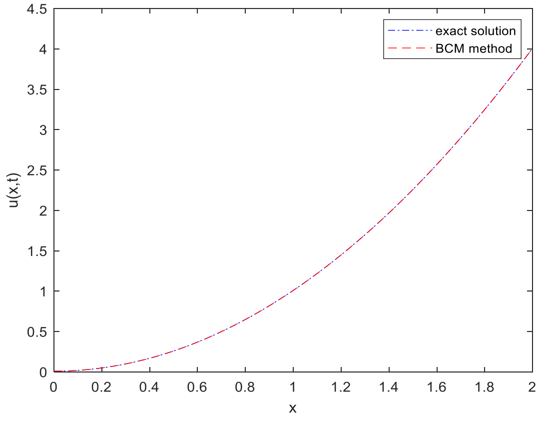

Figure 5: plot of u with respect to x at 0 and 2 where t=1.

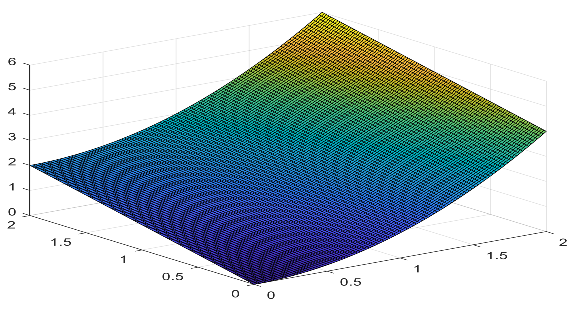
Figure 6 : Depiction of solution u in the domain $x \in [0,2]$ and $t \in [0,2]$.

5. Conclusion

In this work, we have applied BCM to solve telegraph equation. The BCM method applied in three examples whish solved in [5,7], we obtained good numerical results compared the author methods. In fact, the method seemed to be quite simple and stable, the method proves to be applicable and elegant for computer package program.

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